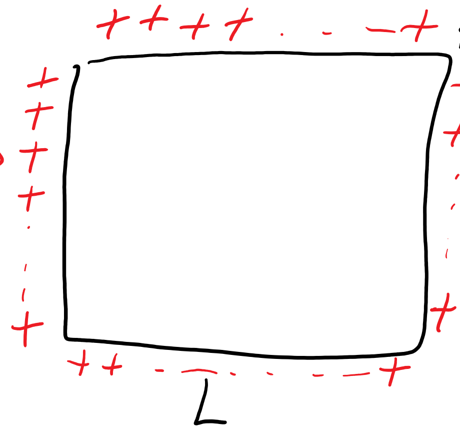
Quantitative estimates for the effects of disorder on low-dimensional lattice systems (one world mathematical physics seminar)

Tuesday, 24 August 2021 16:35

What is the heuristic of Imry-Ma?

Random-field Ising:



$H(\sigma) = -\sum_{u \sim v} \sigma_u \sigma_v - \lambda \sum_v \eta_v \sigma_v$

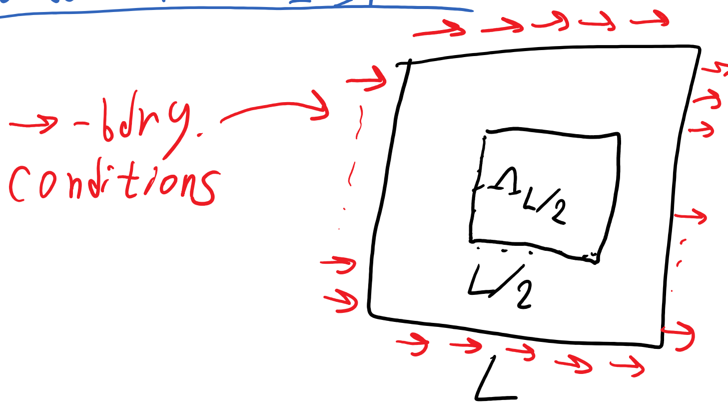
(η_v) IID $N(0,1)$.

Energy of fully + configuration minus energy of fully − configuration $\approx$

$\underbrace{-cL^{d-1}}_{\text{bdry. contribution}} + \lambda L^{d/2} \cdot N(0,1)$

Is this negative? Yes, in $d \geq 3$.
No, in some large domain, in $d = 1, 2$.

Random-field XY: $\sigma_v \in S^1 =$



$H(\sigma) = \sum_{u \sim v} \|\sigma_u - \sigma_v\|^2 - \lambda \sum_v \eta_v \cdot \sigma_v$.

(η_v) IID two-dimensional standard Gaussians.

What is the difference of free energies between the fully → configuration in $\Lambda_{L/2}$ and the fully ← configuration in $\Lambda_{L/2}$?

[illegible]

in $\Lambda_{L/2}$ and the fully $\leftarrow$ configuration in $\Lambda_{L/2}$!

The bdry. contribution reduces to L^{d-2} due to the continuous symmetry.

Heuristically, this causes the system to be disordered, at arbitrary weak field, in dimensions $d \le 4$.

Theorem (Dario-Harel-Peled 2021): For the $\underbrace{\text{spin}}^{\text{random-field}}$ $O(n)$, $n \ge 2$, model in dimensions $d = 2, 3, 4$, for any $L \ge 1$ and any fixed bdry. cond. τ:

$$\mathbb{E}\left\| \frac{1}{L^d} \sum_{v \in \Lambda_{L/2}} \langle \sigma_v \rangle^{\tau}_{\Lambda_L} \right\| \le \begin{cases} \frac{1}{L} & d=2 \\ \frac{1}{\sqrt{L}} & d=3 \\ \frac{1}{\sqrt{\log\log L}} & d=4 \end{cases}$$

Quantitative version of the results of Aizenman-Wehr (1989).

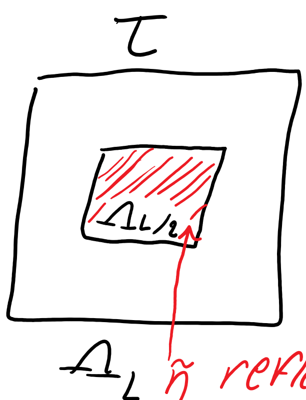
Idea of proof: Fix $n=2$ and discuss only the ground state (for simplicity).

Fix the bdry. cond. τ. Write

$E_L^{\tau,\eta}$ = energy of the ground state $\sigma_L^{\tau,\eta}$ in Λ_L.

Given η, define

$$\tilde{\eta}_v = \begin{cases} \eta_v & v \notin \Lambda_{L/2} \\ -\eta_v & v \in \Lambda_{L/2} \end{cases}$$



$\tilde{\eta}$ reflects random field around y-axis here.

Main quantity:

$$E^{\tau,\eta} - E^{\tau,\tilde{\eta}}$$

Main quantity:

around y-axis here.

$$E_L^{\tau,\eta} - E_L^{\tau,\tilde{\eta}}$$

Claim: $G = \mathbb{E}(E_L^{\tau,\eta} - E_L^{\tau,\tilde{\eta}} \mid \eta_{\Lambda_{L/2}}) \le C L^{d-2}$.

G is a fcn. of $(\eta_v)_{v \in \Lambda_{L/2}}$ obtained by averaging over the other η.

Idea: $\sigma_L^{\tau,\eta}$ is ground state in Λ_L.

Can rotate it slowly between Λ_L and $\Lambda_{L/2}$ so that it is reversed on $\Lambda_{L/2}$, and at the same time rotate the random field η there.

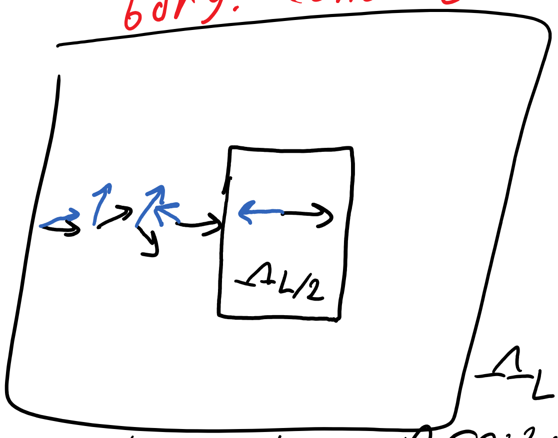
bdry. cond. τ

"spin wave" as in the proofs of the Mermin-Wagner theorem.

Uniform rotation

in black - spins before rotation

in blue - spins after rotation



By averaging the two possibilities of a clockwise and anti-clockwise rotation, the average energy difference is only of order $\frac{1}{L^2}$ per bond. $\square$

Now consider the derivative of $G = G(\eta_{\Lambda_{L/2}})$ with respect to the average field $\hat{\eta} := \frac{1}{|\Lambda_{L/2}|}\sum \eta_v$.

the average field $\hat{\eta}_{L/2} := \frac{1}{|\Lambda_{L/2}|}\sum_{v\in\Lambda_{L/2}} \eta_v$.

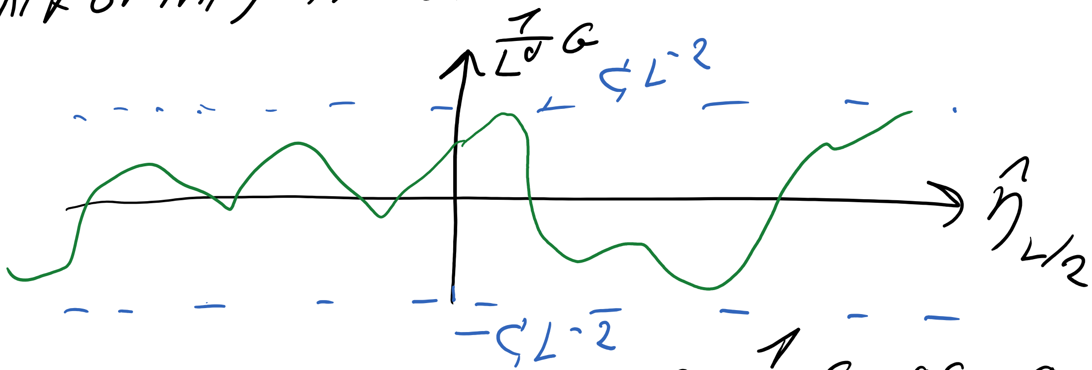
We have $\underbrace{\frac{\partial G}{\partial \hat{\eta}_{L/2}}}_{\text{From the definition of } G} = \mathbb{E}\left(\sum_{v\in\Lambda_{L/2}} \sigma_L^{\tau,\eta}(v) + \sigma_L^{\tau,\tilde{\eta}}(v) \,\Big|\, \eta_{\Lambda_{L/2}}\right)$

since η and $\tilde{\eta}$ are identically distributed

$$\Rightarrow \mathbb{E}\left(\frac{\partial G}{\partial \hat{\eta}_{L/2}}\right) = 2\,\mathbb{E}\left(\sum_{v\in\Lambda_{L/2}} \sigma_L^{\tau,\eta}(v)\right).$$

Thus, the theorem will follow from an upper bound on $\mathbb{E}\left(\frac{\partial G}{\partial \hat{\eta}_{L/2}}\right)$.

We know that $|G| \le C L^{d-2}$ (by above claim) uniformly in the random field η.



We consider the graph of $\frac{1}{L^d}G$ as a fcn. of $\hat{\eta}_{L/2}$, for fixed values of the other degrees of freedom of η.
As $\hat{\eta}_{L/2}$ fluctuates of the scale $L^{-d/2}$ while $\frac{1}{L^d}|G| \le C L^{-2}$ for all η, we conclude that the average slope of $\frac{1}{L^d}\cdot G$ is not too large. Specifically we get $\mathbb{E}\left(\frac{\partial G}{\partial \hat{\eta}}\right) \le C\frac{L^{d/2}}{L^2} \le \begin{cases} \frac{1}{L} & d=2 \end{cases}$

get $\mathbb{E}\left(\frac{\partial G}{\partial \eta_{L/2}}\right) \leq C \frac{L^{d/2}}{L^2} \leq \begin{cases} \frac{1}{L} & d=2 \\ \frac{1}{\sqrt{L}} & d=3 \\ C & d=4 \end{cases}$

Which proves the theorem when $d=2,3$.

In dimension $d=4$, the analysis is refined using a recursive Mandelbrot percolation structure.